MOTION OF A CHARGED PARTICLE IN CROSSED FIELDS WHEN THE MAGNETIC FIELD IS STRONGLY INHOMOGENEOUS

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We discuss the astrophysical problem [1] of the motion of a charged particle in crossed fields. In contrast with previous treatments [2-5] we allow strong inhomogeneities of the magnetic field.

1. Equations of Motion and an Adiabatic Invariant. In a number of physically interesting cases it can be assumed that a moving particle does not interact with a medium (plasma) in which currents are flowing. We denote the radius vector to the particle by $\mathbf{r}$ its mass by m, and its charge by $e > 0$; $\mathbf{E}$ and $\mathbf{H}$ are respectively the intensities of the electric and magnetic fields. The equation of motion has the form

$$\mathbf{r}^{\cdot\cdot} = \frac{e}{m}\mathbf{E} + \frac{e}{mc}\mathbf{r}^{\cdot}\times\mathbf{H}. \tag{1.1}$$

In the case under consideration $\mathbf{E} = E\mathbf{j}$, $E = \mathrm{const} > 0$, $\mathbf{H} = H_z(x)\mathbf{k}$, $H_z(x) > 0$, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors along the coordinates axes. The magnetic field is produced by currents along the y axis; the current density depends on x only. The cases $e < 0$, $H_z < 0$, etc., can be treated similarly.

We denote characteristic values of the magnetic field intensity, the velocity component in the xy plane, and the Larmor frequency and radius respectively by [H], [v], λ and R_L. We introduce the dimensionless time λt, and the dimensionless coordinates x/R_L and y/R_L. Retaining the previous notation for the dimensionless quantities, and projecting (1.1) onto the x and y axes, we obtain

$$x^{\cdot} = u,\ u^{\cdot} = h(x)v,\ v^{\cdot} = \varepsilon - h(x)u, \tag{1.2}$$

where $v = y^{\cdot}$; $\varepsilon = Ec/[v][H]$; $h(x) = H_z(x)/[H]$.

Motion along the z axis is uniform and of no further interest.

As in [2-5] it is assumed that $\varepsilon \ll 1$, but in contrast with those papers the function h(x) is not assumed slowly varying.

We replace v by a new variable $\sigma = v + A(x)$, where $dA/dx = h(x)$ and $A(0) = 0$. We obtain from (1.2) $\sigma^{\cdot} = \varepsilon$ and $\sigma = \sigma_0 + \tau$, where $\tau = \varepsilon(t - t_0)$. The first two equalities of Eqs. (1.2) take the form

$$x^{\cdot} = u,\ u^{\cdot} = h(x)(\sigma - A(x)). \tag{1.3}$$

Equations (1.3) describe the uniform motion of an arbitrary particle in a force field which varies slowly with time and has the potential

$$\Pi = (1/2)(\sigma - A(x))^2.$$

The function A(x) is strictly monotonic, and therefore the phase trajectories of system (1.3) for $\sigma = \mathrm{const}$ are closed. The motion of an arbitrary particle is vibrational with slowly varying amplitude, period, and position of the center. Except for small quantities, the center of vibrations x_c is the root of the equation $A(x_c) = \sigma$; the amplitude is $x_2 - x_1$, where x_1, x_2, $x_1 < x_2$ are roots of the equation $(\sigma - A(x))^2 = 2H$, and H is the Hamiltonian of an arbitrary particle

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$$H = \frac{1}{2}u^2 + \frac{1}{2}(\sigma - A(x))^2. \tag{1.4}$$

System (1.3) has the adiabatic invariant [6]

$$I(H, \sigma) = \frac{1}{\pi}\int_{x_1}^{x_2} \sqrt{2H - (\sigma - A(\xi))^2}\, d\xi. \tag{1.5}$$

The integral in (1.5) and the one in (1.8) are evaluated for constants σ and H; the quantities x_1 and x_2 are functions of σ and H.

For $t \leqslant t_0 + T/\varepsilon$ the quantity I is conserved, except for small quantities $O(\varepsilon)$. This enables us to investigate the change of other dynamical characteristics of the particle. Such an investigation is performed later, since the invariant in the present problem is different from the familiar form in being "nonlocal."

By solving the equation $I = I(H, \sigma)$ for H, we obtain the relation $H = H(I, \sigma)$ which determines the change in H in the first approximation in the form $H(t_1) - H(t_0) = H(I, \sigma_1) - H(I, \sigma_0)$. Expressing $\omega = \sigma H/\sigma I$ in terms of I and σ, we find the change in the period $2\pi/\omega$, and from the relations given above, the evolution of x_c, x_1, x_2.

Thus, the determination of the slow evolution is reduced to the inversion of (1.5) and not to the integration of the averaged equations, as is necessary when an adiabatic invariant is not known.

Let us consider the projection of the particle trajectory on the xy plane. The y coordinate of the particle is given by

$$y = y_0 + \int_{t_0}^{t} (\sigma - A(x))\, dt. \tag{1.6}$$

The function y(t) is the superposition of fast vibrations on a rapid, almost uniform motion. During a time of the order of $1/\varepsilon$ the particle is displaced a distance $\sim R_L$ along the x axis, and a large distance $\sim R_L/\varepsilon$ along the y axis. Therefore the projection of the particle trajectory on the xy plane is a spiral with the "axis" slightly inclined to the y axis. Since, in contrast with known problems, the velocity of the center of a turn along the y axis is sizable, the spiral will be elongated; its turns are far apart.

It follows from (1.3) that

$$\dot{H} = \varepsilon(\sigma - A(x)) = \varepsilon v, \quad H - H_0 = \varepsilon(y - y_0). \tag{1.7}$$

This expresses the obvious fact that the increase in the kinetic energy of the particle is equal to the work done by the electrostatic forces. By using (1.7), which contains information on the function H, it is possible to obtain knowledge of the trajectory.

It is reasonable to regard the point

$$x = x_c(\tau), \quad y = y_c(\tau) = [H(I, \tau) - H_0]/\varepsilon$$

as the center of a turn.

The relations $A(x_c) = \sigma$ and $\dot{x}_c = \varepsilon/h(x_c)$ determine a slow displacement of the center in the positive direction of the x axis. The displacement along the y axis is determined by the change in H. Therefore the construction of the trajectory of the center of a turn is also reduced to the inversion of integral (1.5).

The magnetic field intensity h(x) and the potential A(x) do not enter Eq. (1.5) directly. It might seem that the dynamical characteristics of the particle depend not only on the values of h(x) on the turn being considered, but also on the values of h(x) on the parts of the path already transversed by the particle. We show that this is not so. To do this we compare the motions of two particles in the fields which differ only in a certain interval $[x_a, x_b]$; i.e., we compare motions in the fields h(x) and $h(x) + \Delta(x)$, where $\Delta \neq 0$ only for

$x \in [x_a, x_b]$.

Suppose both particles begin to move under identical initial conditions, and do not enter the region $x > x_a$ during the first turns. Along these first turns the particle motions and the values of I will be identical. Along the turns where $x > x_a$ the potential $A(x)$ for the second particle will be different from that for the first by the constant term

$$\Delta A = \int_{x_a}^{x_b} \Delta(\xi)\, d\xi.$$

We introduce a new argument $t^{(2)} = t - \Delta A/\varepsilon$ into Eqs. (1.3) for the second particle. As a result Eqs. (1.3) for the first particle in the variables $x^{(1)}$, $u^{(1)}$, t are the same as for the second particle in the variables $x^{(2)}$, $u^{(2)}$, $t^{(2)}$ (the superscript denotes the number of the particle). The values of I will also be identical for both particles. Consequently, except for small quantities, the dynamical characteristics of both particles for $x > x_b$ will be identical, i.e., $H^{(1)}(\tau) = H^{(2)}(\tau - \Delta A)$, $\omega^{(1)}(\tau) = \omega^{(2)}(\tau - \Delta A)$, etc. The only difference will be in the transit times of the particles near the same point.

We now analyze qualitatively the change in kinetic energy during the motion of a particle. Suppose $dh/dx > 0$. Then

$$\frac{\partial I(H,\sigma)}{\partial \sigma} = -\frac{1}{\pi}\int_{x_1}^{x_2} \frac{\sigma - A(\xi)}{\sqrt{2H - (\sigma - A(\xi))^2}}\, d\xi =$$

$$= -\frac{2H}{\pi}\int_0^{\pi/2} \frac{h(A^{-1}(\sigma + \sqrt{2H}\sin\theta)) - h(A^{-1}(\sigma - \sqrt{2H}\sin\theta))}{h(A^{-1}(\sigma + \sqrt{2H}\sin\theta))\, h(A^{-1}(\sigma - \sqrt{2H}\sin\theta))} \sin\theta\, d\theta < 0, \tag{1.8}$$

i.e., for H = const the action $I(H, \sigma)$ decreases with time. But since I is conserved, the kinetic energy increases. Similarly, if $dh/dx < 0$ the kinetic energy decreases with time.

2. Comparison with Problems of Motion in a Weakly Inhomogeneous Field. In this problem it is assumed [2-5] that $R_L \sim \varepsilon[r]$, where [r] is the characteristic scale of variation of the magnetic field. The problem considered above pertains to the case $R_L \sim [r]$. We compare the solutions of these two problems in order to find how strong inhomogeneities of a magnetic field affect the motion of particles in this case.

We consider first the problem of motion in a weakly inhomogeneous field which now has singularities as compared with [2-5]. From the known [2-5] equations we obtain

$$\rho^{\cdot} = \varepsilon p \cos\psi,\quad p^{\cdot} = -\varepsilon \sin\psi,\quad \psi^{\cdot} = h(\rho) - \varepsilon\cos\psi/p, \tag{2.1}$$

where $\rho = \varepsilon x$; $p = (u^2 + v^2)^{1/2}$; $\cos\psi = u/p$; $\sin\psi = -v/p$.

Averaging over ψ leads to the trivial result that in the first approximation ρ and p are conserved for times $\sim 1/\varepsilon$; therefore the conclusion that the known adiabatic invariant p^2/h is conserved is also trivial. These conclusions are insufficient, since it is necessary to trace the motion of a particle until h is not significantly changed, i.e., over distances $\sim[r]$ for times $\sim 1/\varepsilon^2$.

In view of the above it is necessary to find solutions of Eqs. (2.1) in the second approximation. We obtain the following equations for the drift components ξ and η of the functions ρ and p in the second approximation:

$$\frac{d\xi}{d\psi} = \varepsilon^2 \frac{1}{h^2(\xi)},\quad \frac{d\eta}{d\psi} = \frac{1}{2}\varepsilon^2 \frac{dh/d\xi}{h^3(\xi)}\eta.$$

These equations have the integral $\eta^2/h(\xi)$ = const. Hence it follows that p^2/h is an adiabatic invariant.

In the present case an adiabatic invariant of known form was obtained. However, this is not obvious a priori, since in a nondegenerate problem an adiabatic invariant is obtained from first order terms, and in a different way. The fact that the invariant is conserved for times $\sim 1/\varepsilon^2$ is also an important feature. A similar situation arises when $v_\parallel = 0$ during the whole time of motion [5].

We now compare the motions of two particles, one of which moves in a weakly inhomogeneous field at all times, and the other begins and ends its motion in the same regions of the field as the first particle, but with part of its path through a strong inhomogeneity. Suppose the particles begin their motions with the same initial conditions. At the start of the motion both particles have the invariant p^2/h. This invariant is conserved for the first particle during the whole time of the motion. When the second particle enters the region of the strongly inhomogeneous field the quantity p^2/h will not be conserved, but the invariant (1.5) will be conserved. But in the region of the weakly inhomogeneous field we have, except for small quantities,

$$A(\xi) = A(x_c) + h(x_c)(\xi - x_c),$$

$$I(H, \sigma) = \frac{1}{\pi}\int_{x_1}^{x_2} \sqrt{2H - h^2(x_c)(\xi - x_c)^2}\,d\xi = p^2/h(x_c),$$

i.e., the invariant (1.5) goes over into p^2/h. Consequently, in entering the strongly inhomogeneous field the value of I will be determined by the value of p^2/h in the weakly inhomogeneous field, and vice versa. Since the second particle finally enters the same region of the weakly inhomogeneous field as the first particle, it will have the same value (initial) of the invariant p^2/h.

Consequently, the presence of jumps in the magnetic field intensity under the conditions considered does not affect the dynamical characteristics of the motion. In other words, in the cases considered there is no "superadiabatic" [1] acceleration of particles, i.e., larger than is observed when the adiabatic invariant p^2/h is conserved.

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